Preliminary version

DILOGARITHM IDENTITIES, PARTITIONS AND SPECTRA IN CONFORMAL FIELD THEORY, I

ANATOL N. KIRILLOV*

Isaac Newton Institute for Mathematical Sciences,
20 Clarkson Road, Cambridge, CB3 OEH, U.K.
and
Steklov Mathematical Institute,
Fontanka 27, St.Petersburg, 191011, Russia

ABSTRACT

We prove new identities between the values of Rogers dilogarithm function and describe a connection between these identities and spectra in conformal field theory (part I). We also describe the connection between asymptotical behaviour of partitions of some class and the identities for Rogers dilogarithm function (part II).

Introduction.

The dilogarithm function $Li_2(x)$ defined for $0 \le x \le 1$ by

$$Li_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t} dt,$$

is one of the lesser transcendental functions. Nonetheless, it has many intriguing properties and has appeared in various branches of mathematics and physics such as number theory (the study of asymptotic behaviour of partitions, e.g. [RS], [AL]; the values of ζ-functions in some special points [Za]) and algebraic K-theory (the Bloch group and a torsion in $K_3(\mathbf{R})$ - A.Beilinson, S.Bloch [Bl], A.Goncharov), geometry of hyperbolic three-manifolds [Th], [NZ], [Mi], [Y], representation theory of Virasoro and Kac-Moody algebras [Ka], [KW], [FS] and conformal field theory (CFT).

In physics, the dilogarithm appears at first from a calculation of magnetic susceptibility in the XXZ model at small magnetic field [KR1], [KR2], [Ki1], [BR]. More recently [Z], the dilogarithm identities (through the Thermodynamic Bethe Ansatz (TBA)) appear in the context of investigation of UV limit or the critical behaviour of integrable 2-dimensional quantum field theories and lattice models [Z],

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[DR], [K-M], [KBP], [Kl], [KP],...). Evenmore, it was shown (e.g. [NRT]) using a method of Richmond and Szekeres [RS], that the dilogarithm identities may be derived from an investigation of the asymptotic behaviour of some characters of $2d$ CFT. Thus, it seems a very interesting problem to lift the dilogarithm identity in question to some identity between the characters of certain conformal field theory. A partial solution of this problem (without any proofs!) contained in [Te] and [KKMM].

One aim of this paper (part I) is to prove some new identities between the values of Rogers dilogarithm function using the analytical methods (see [Le], [Ki1]). Based on such identities we give an answer on one question of W.Nahm [Na]: how big may be the following abelian group

$$\mathcal{W} := \left\{ \sum_i \frac{n_i L(\alpha_i)}{L(1)} \mid n_i \in \mathbf{Z}, \quad \alpha_i \in \overline{\mathbf{Q}} \cap \mathbf{R} \ \text{ for all } i \right\} \cap \mathbf{Q} \ \ ?$$

Theorem ♣ . The abelian group $\mathcal{W}$ coincides with $\mathbf{Q}$, i.e. any rational number may be obtained as the value of some dilogarithm sum.

A proof of Theorem ♣ follows from Proposition 4.5. We also give a proof (and the different generalizations) of an identity (3.1) from [NRT] (see our Theorem 3.1 and Proposition 5.4). Note the following “reciprosity law” for dilogarithm sums (see Theorem 3.2)

$$s(j,n,r) + s(j,r,n) = nr - 1.$$

This is a consequence of the corresponding reciprosity law for the Dedekind sums [Ra]. Note also that the author don't know any CFT interpretation for the dilogarithm identities from Propositions 4.4 and 5.4.

Part II will be deal with the partition identities which”materialise” the dilogarithm ones.

§1. Definition and the basic properties of Rogers dilogarithm.

Let us remind the definition of the Rogers dilogarithm function $L(x)$ for $x \in (0,1)$

$$L(x) = -\frac{1}{2}\int_0^x \left[\frac{\log(1-x)}{x} + \frac{\log x}{1-x}\right] dx = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2}\log x \cdot \log(1-x). \qquad (1.1)$$

The following two classical results (see e.g. [Le2], [GM], [Ki1]) contain the basic properties of the function $L(x)$.

Theorem A. The function $L(x) \in C^{\infty}((0,1))$ and satisfies the following functional equations

$$1. \ \ L(x) + L(1-x) = \frac{\pi^2}{6}, \quad 0 < x < 1, \qquad (1.2)$$

$$2. \ \ L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right), \qquad (1.3)$$

where $0 < x, \ y < 1$.

Theorem B. Let $f(x)$ be a function of class $C^3((0,1))$ and satisfies the relations (1.2) and (1.3). Then we have

$$f(x) = \text{const} \cdot L(x)$$

We continue the function $L(x)$ on all real axis $\mathbf{R} = \mathbf{R}^1 \cup \{\pm\infty\}$ by the following rules

$$L(x) = \frac{\pi^2}{3} - L(x^{-1}), \quad \text{if} \quad x > 1, \tag{1.4}$$

$$L(x) = L\left(\frac{1}{1-x}\right) - \frac{\pi^2}{6}, \quad \text{if} \quad x < 0, \tag{1.5}$$

$$L(0) = 0, \quad L(1) = \frac{\pi^2}{6}, \quad L(+\infty) = \frac{\pi^2}{3}, \quad L(-\infty) = -\frac{\pi^2}{6}.$$

The present work will concern with relations between the values of the Rogers dilogarithm function at certain algebraic numbers. More exactly, let us consider an abelian subgroup $\mathcal{W}$ in the field of rational numbers $\mathbf{Q}$

$$\mathcal{W} = \left\{ \sum_i \frac{n_i L(\alpha_i)}{L(1)} \mid n_i \in \mathbf{Z}, \quad \alpha_i \in \overline{\mathbf{Q}} \cap \mathbf{R} \ \text{ for all } \ i \right\} \cap \mathbf{Q}. \tag{1.6}$$

According to a conjecture of $\mathcal{W}$. Nahm [Na] the abelian group $\mathcal{W}$ "coincides" with the spectra in rational conformal field theory. Thus it seems very interesting task to obtain more explicit description of the group $\mathcal{W}$ (e.g. to find a system of generators for $\mathcal{W}$) and also to connect already known results about the spectra in conformal field theory (see e.g. [BPZ], [FF], [GKO], [FQS], [Bi], [Ka], [KP]) with suitable elements in $\mathcal{W}$. One of our main results of the present paper allows to describe some part of a system of generators for abelian group $\mathcal{W}$. As a corollary, we will show that the spectra of unitary minimal models [BPZ], [GKO] and some others are really contained in $\mathcal{W}$. But at first we remind some already known relations between the values of the Rogers dilogarithm function.

§2. Some dilogarithm relations.

It is easy to see from (1.2) and (1.3) that

$$L(\frac{1}{2}) = \frac{\pi^2}{12}, \quad L(\frac{1}{2}(\sqrt{5}-1)) = \frac{\pi^2}{10}, \quad L(\frac{1}{2}(3-\sqrt{5})) = \frac{\pi^2}{15}. \tag{2.1}$$

Proof. Let us put $\alpha := \frac{1}{2}(\sqrt{5}-1)$. It is clear that $\alpha^2 + \alpha = 1$. So we have $L(\alpha^2) = L(1-\alpha) = L(1) - L(\alpha)$. Now we use the Abel's formula

$$L(x^2) = 2L(x) - 2L\left(\frac{x}{1+x}\right), \tag{2.2}$$

which may be obtained from (1.3) in the case $x=y$. From (2.2) we find that

$$L(\alpha^2)=2L(\alpha)-2L(\frac{\alpha}{1+\alpha})=2L(\alpha)-2L(\alpha^2).$$

So, $3L(\alpha^2)=2L(\alpha)$. But as we already saw,

$$L(1)=L(\alpha)+L(\alpha^2)=\frac{5}{3}L(\alpha).$$

This proves (2.1).

■

Apparently, there are no other algebraic point from the interval $(0,1)$ at which there is such an elementary evaluation of Rogers dilogarithm function. However, there are many identities relating the values of dilogarithm function at various powers of algebraic numbers. It is interesting to write some of this identities in order to compare the elements of the abelian group W obtained by such manner with the spectra of known conformal models. As concerning identities between the values of dilogarithm function, we follow [Le1], [Le2], [Lo] and [RS].

$$6L(\frac{1}{3})-L(\frac{1}{9})=\frac{\pi^2}{3}, \tag{2.3}$$

$$\sum_{k=2}^{n} L\left(\frac{1}{k^2}\right)+2L\left(\frac{1}{n+1}\right)=\frac{\pi^2}{6}, \tag{2.4}$$

consequently,

$$\sum_{k=2}^{\infty} L\left(\frac{1}{k^2}\right)=\frac{\pi^2}{6}.$$

This identities may be easily deduced from (1.2) and (2.2). Again, if $\alpha=\sqrt{2}-1$, we have the relations

$$4L(\alpha)+L(1-\alpha^2)=\frac{5\pi^2}{12},$$

$$4L(\alpha)+4L(\alpha^2)+L(1-\alpha^4)=\frac{7\pi^2}{12} \tag{2.5}$$

Proof. We use Abel's formula (2.2) and relation (1.3) with $y=\frac{1}{2}$

$$L(\frac{1}{2})+L(x)=L\left(\frac{x}{2}\right)+L\left(\frac{x}{2-x}\right)+L\left(\frac{1-x}{2-x}\right). \tag{2.6}$$

We have

$$L(\alpha^2)=2L(\alpha)-2L\left(\frac{\alpha}{1+\alpha}\right)=2L(\alpha)-2L\left(\frac{2-\sqrt{2}}{2}\right),$$

$$L(\frac{1}{2})=2L(\frac{\sqrt{2}}{2})-2L(\alpha)=\frac{\pi^2}{3}-2L(\frac{2-\sqrt{2}}{2})-2L(\alpha),$$

$$L(\alpha^4)=2L(\alpha^2)-2L\left(\frac{\alpha^2}{1+\alpha^2}\right)=2L(\alpha^2)-2L\left(\frac{2-\sqrt{2}}{4}\right),$$

$$L\left(\frac{2-\sqrt{2}}{2}\right)+L(\frac{1}{2})=L\left(\frac{2-\sqrt{2}}{4}\right)+L(\alpha)+L(\alpha^2).$$

Excluding successively $L(\frac{2-\sqrt{2}}{2})$ and $L(\frac{2-\sqrt{2}}{4})$ from these relations we obtain (2.5). ∎

Watson [Wa] found three relations involving the roots of the cubic x^3+2x^2-x-1. Namely, if we take $\alpha = \dfrac{1}{2}\sec\dfrac{2\pi}{7}$, $\beta = \dfrac{1}{2}\sec\dfrac{\pi}{7}$, $\gamma = 2\cos\dfrac{3\pi}{7}$, then α, $-\beta$ and $-\dfrac{1}{\gamma}$ are the roots of this cubic and

$$\begin{aligned} L(\alpha) + L(1-\alpha^2) &= \frac{4\pi^2}{21}, \\ 2L(\beta) + L(\beta^2) &= \frac{5\pi^2}{21}, \\ 2L(\gamma) + L(\gamma^2) &= \frac{4\pi^2}{21}. \end{aligned} \tag{2.7}$$

Lewin [Le1] and Loxton [Lo] found three relations involving the roots of the cubic x^3+3x^2-1. Namely, if we take $\delta = \dfrac{1}{2}\sec\dfrac{\pi}{9}$, $\epsilon = \dfrac{1}{2}\sec\dfrac{2\pi}{9}$, $\zeta = 2\cos\dfrac{4\pi}{9}$, then δ, $-\epsilon$ and $-\dfrac{1}{\zeta}$ are the roots of this cubic and

$$\begin{aligned} 3L(\delta) + 3L(\delta^2) + L(1-\delta^3) &= \frac{17\pi^2}{18}, \\ 6L(\epsilon) + 9L(1-\epsilon^2) + 2L(1-\epsilon^3) + L(\epsilon^6) &= \frac{31\pi^2}{18}, \\ 6L(\zeta) + 9L(1-\zeta^2) + 2L(1-\zeta^3) + L(\zeta^6) &= \frac{35\pi^2}{18}. \end{aligned} \tag{2.8}$$

§3. Basic identities and conformal weights.

In this section we present our main results dealing with a computation of the following dilogarithm sum

$$\sum_{k=1}^{n-1}\sum_{m=1}^{r} L\left(\frac{\sin k\varphi\cdot\sin(n-k)\varphi}{\sin(m+k)\varphi\cdot\sin(m+n-k)\varphi}\right) := \frac{\pi^2}{6}s(j,n,r), \tag{3.1}$$

where $\varphi = \dfrac{(j+1)\pi}{n+r}$, $0 \le j \le n+r-2$.

It is clear that $s(j,n,r) = s(n+r-2-j,n,r)$, so we will assume in sequel that $0 \le 2j \le n+r-2$.

The dilogarithm sum (3.1) corresponds to the Lie algebra of type A_{n-1}. The case $j=0$ was considered in our previous paper [Ki1], where it was proved that $s(0,n,r) = \dfrac{(n^2-1)r}{n+r}$. It was stated in [Ki1] that this number coincides with the central charge of the $SU(n)$ level r WZNW model. Before formulating our result about computation of the sum $s(j,n,r)$ let us remind the definition of Bernoulli polynomials. They are defined by the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad |t|<2\pi.$$

We use also modified Bernoulli polynomials

$$\overline{B}_n(x)=B_n(\{x\}),\quad \text{where}\quad \{x\}=x-[x]$$

be a fractional part of $x\in\mathbf{R}$. It is well-known that

$$\begin{aligned}\overline{B}_{2n}(x)&=(-1)^n\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^{2n}},\\ \overline{B}_{2n+1}(x)&=(-1)^{n-1}\frac{2(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\frac{\sin 2k\pi x}{k^{2n+1}}.\end{aligned}\tag{3.2}$$

Theorem 3.1. We have

$$s(j,n,r)=6(r+n)\sum_{k=0}^{[\frac{n-1}{2}]}\left\{\frac{1}{6}-\overline{B}_2\Big((n-1-2k)\theta\Big)\right\}-\frac{1}{4}\Big\{2n^2+1+3(-1)^n\Big\},\tag{3.3}$$

where $\theta=\dfrac{j+1}{r+n}$ and g.c.d.$(j+1,r+n)=1$.

Theorem 3.2. (level-rank duality [SA], [KN1])

$$s(j,n,r)+s(j,r,n)=nr-1.\tag{3.4}$$

Corollary 3.3. We have

$$s(j,n,r)=c_r^{(n)}-24h_j^{(r,n)}+6\cdot\mathbf{Z}_+,\tag{3.5}$$

where

$$c_r^{(n)}=\frac{(n^2-1)r}{n+r},\quad h_j^{(r,n)}=\frac{n(n^2-1)}{24}\cdot\frac{j(j+2)}{r+n},\quad 0\le j\le r+n-2,\tag{3.6}$$

are the central charge and conformal dimensions of the $SU(n)$ level r WZNW primary fields, respectively.

Proof. Let us remind that

$$B_2(x)=x^2-x+\frac{1}{6}\quad\text{and}\quad \overline{B}_2(x)=B_2(\{x\}).$$

Thus,

$$
\begin{aligned}
&6(r+n)\sum_{k=0}^{[\frac{n-1}{2}]}\left(\frac{1}{6}-\overline{B}_2\Big((n-1-2k)\theta\Big)\right)=\\
&=6(r+n)\sum_{k=0}^{[\frac{n-1}{2}]}(n-1-2k)\theta-6(r+n)\sum_{k=0}^{[\frac{n-1}{2}]}(n-1-2k)^2\theta^2+\\
&+6(r+n)\sum_{k=0}^{[\frac{n-1}{2}]}\Big[(n-1-2k)\theta\Big]\left((n-1-2k)\theta-1+\Big\{(n-1-2k)\theta\Big\}\right):=\\
&=6\Sigma_1-6\Sigma_2+6\Sigma_3.
\end{aligned}
$$

Now if we take $\theta=\dfrac{j+1}{r+n}$, then it is clear that $\Sigma_3\in\mathbf{Z}_+$. In order to compute Σ_1 and Σ_2 we use the following summation formulae

$$
\begin{aligned}
&\sum_{k=0}^{[\frac{n-1}{2}]}(n-1-2k)=\frac{2n^2-1+(-1)^n}{8}=\left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right],\\
&\sum_{k=0}^{[\frac{n-1}{2}]}(n-1-2k)^2=\frac{n(n^2-1)}{6}.
\end{aligned}
$$

Consequently $\quad s(j,n,r)=$

$$
\begin{aligned}
&=\frac{3(2n^2-1+(-1)^n)(j+1)}{4}-\frac{n(n^2-1)(j+1)^2}{r+n}-\frac{2n^2+1+3(-1)^n}{4}+6\Sigma_3=\\
&=\frac{(n^2-1)r}{r+n}-\frac{n(n^2-1)j(j+2)}{r+n}+6j\left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right]+6\Sigma_3.
\end{aligned}
$$

■

For small values of j we may compute the sum in (3.3) and thus to find corresponding positive integer in (3.5).

Corollary 3.4.
i) if $j\le r$, then

$$
s(j,2,r)=c_r^{(2)}-24h_j^{(r,2)}+6j=\frac{3r}{r+2}+6\,\frac{j(r-j)}{r+2}. \tag{3.7}
$$

ii) if $2j\le r+1$, then

$$
s(j,3,r)=\frac{8r}{r+3}-24\,\frac{j(j+2)}{r+3}+12j. \tag{3.8}
$$

iii) if $(n-1)j<r+1$, then

$$
s(j,n,r)=c_r^{(n)}-24h_j^{(r,n)}+6j\cdot\left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right]. \tag{3.9}
$$

Proof. An assumption $(n-1)j < r+1$ is equivalent to a condition $\dfrac{(n-1)(j+1)}{n+r} < 1$. So the term Σ_3 (see a proof of Corollary 3.3) is equal to zero.

■

It seems interesting to find a meaning of the positive integer in (3.5). Now we want to find a "dilogarithm interpretation" of the central charges and conformal dimensions for some well-known conformal models.

Corollary 3.5. We have

$$s(j_1,2,k)+s(0,2,1)-s(j_2,2,k+1)=c_k-24h_{j_1+1,j_2+1}+6(j_1-j_2)(j_1-j_2+1), \quad (3.10)$$

where

$$c_k=1-\frac{6}{(k+2)(k+3)}, \quad h_{r,s}^{(k)}=\frac{[(k+3)r-(k+2)s]^2-1}{4(k+2)(k+3)} \quad (3.11)$$

are the central charge and conformal dimensions of the primary fields for unitary minimal conformal models [BPZ], [Ka], [GKO].

Corollary 3.6.

$$s(j_1,n,k)+s(0,n,1)-s(j_2,n,k+1)=c_{k,n}-24h_{j_1+1,j_2+1}^{(n)}+12\mathbf{Z}_+, \quad (3.12)$$

where

$$\begin{aligned} c_{k,n}&=(n-1)\left\{1-\frac{n(n+1)}{(k+n)(k+n+1)}\right\},\\ h_{r,s}^{(n)}(k)&=\frac{n(n^2-1)}{24}\cdot\frac{[(k+n+1)r-(k+n)s]^2-1}{(k+n)(k+n+1)} \end{aligned} \quad (3.13)$$

are the central charge and conformal dimensions of the primary fields for W_n models [Bi], [CR].

Corollary 3.7.

$$\begin{aligned} &s(j_1,2,k)+s\left(\frac{1}{2}(1-(-1)^{j_1-j_2}),2,2\right)-s(j_2,2,k+2)=\\ &=c(k)-24\widetilde{h}_{j_1+1,j_2+1}+12\left[\frac{j_1-j_2+1}{2}\right]\cdot\left[\frac{j_1-j_2+2}{2}\right], \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} c(k)&=\frac{3}{2}\left(1-\frac{8}{(k+2)(k+4)}\right),\\ \widetilde{h}_{r,s}:=\widetilde{h}_{r,s}(k)&=\frac{[(k+4)r-(k+2)s]^2-4}{8(k+2)(k+4)}+\frac{1-(-1)^{r-s}}{32}, \end{aligned} \quad (3.15)$$

are the central charge and conformal dimensions of the primary fields for unitary minimal $N=1$ superconformal models [GKO], [MSW].

We give a generalization of Corollaries (3.5)-(3.7) to the case of non-unitary minimal models.

Corollary 3.8. If $p \geq q \geq 2$, then

$$\begin{aligned}&(p-q)s(j_1,2,q-2)+s(0,2,1)-(p-q)s(j_2,2,p-2)=\\&=c-24h_{j_1+1,j_2+1}+6(j_1-j_2)(p-q+j_1-j_2),\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}&c=1-\frac{6(p-q)^2}{pq}, \\ &h_{r,s}:=h_{r,s}(c)=\frac{(pr-qs)^2-(p-q)^2}{4pq}, \quad r<q,\ s<p,\end{aligned} \tag{3.17}$$

are the central charge and conformal dimensions of the primary fields for non-unitary (if $p-q \geq 2$) Virasoro minimal models [FQS]. Note that "remainder term" in (3.16)

$$6(j_1-j_2)(p-q+j_1-j_2)$$

appears to be positive for all $0 \leq j_1 < q, \quad 0 \leq j_2 < p$ iff $p-q=0$ (trivial case) or $p-q=1$ (unitary case).

Corollary 3.9. Let $p \geq q \geq 2$ and $p-q \equiv 0 \pmod 2$. Then

$$\begin{aligned}&\frac{p-q}{2}s(j_1,2,q-2)+s\left(\frac{1}{2}(1-(-1)^{j_1-j_2}),2,2\right)-\frac{p-q}{2}s(j_2,2,p-2)=\\&=\widetilde{c}-24\widetilde{h}_{j_1+1,j_2+1}+6\left\{\frac{(j_1-j_2)(j_1-j_2+p-q)}{2}+\frac{1-(-1)^{j_1-j_2}}{4}\right\},\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}&\widetilde{c}=\frac{3}{2}\left(1-\frac{2(p-q)^2}{pq}\right), \\ &\widetilde{h}_{r,s}=\frac{(pr-qs)^2-(p-q)^2}{8pq}+\frac{1-(-1)^{r-s}}{32}, \quad r<q,\ s<p,\end{aligned} \tag{3.19}$$

are the central charge and conformal dimensions of primary fields for non-unitary (if $p-q>2$) Neveu-Schwarz (if $r-s$ even) or Ramond (if $r-s$ odd) minimal models [FQS].

Corollary 3.10. Let $p \geq q \geq n$, then

$$\begin{aligned}&(p-q)s(j_1,n,q-n)+s(0,n,1)-(p-q)s(j_2,n,p-n)=\\&=c-24h_{j_1+1,j_2+1}(c)+6\mathbf{Z},\end{aligned} \tag{3.20}$$

where

$$c = (n-1)\left\{1 - \frac{n(n+1)(p-q)^2}{pq}\right\},$$
$$h_{rs}(c) = \frac{n(n^2-1)}{24}\frac{(pr-qs)^2-(p-q)^2}{pq}, \quad r<q, \ s<p, \tag{3.21}$$

are the central charge and conformal dimensions of primary fields for non-unitary (if $p-q \geq 2$) W_n minimal models [Bi].

Finally we give a "dilogarithm interpretation" for the central charges and conformal weights of restricted solid-on-solid (RSOS) lattice models and their fusion hierarchies [KP].

Corollary 3.11. We have

$$\begin{aligned} &s(l,2,N) + s(N-1,2,N-2) - s(m-1,2,N-2) = \\ &= c+1-24\Delta+6(l-|m|), \quad m\in\mathbf{Z}, \quad 0\leq l\leq N, \end{aligned} \tag{3.22}$$

where

$$c = \frac{2(N-1)}{N+2} \quad \text{and} \quad \Delta = \frac{l(l+2)}{4(N+2)} - \frac{m^2}{4N} \tag{3.23}$$

are the central charge and conformal weights of $\mathbf{Z}_N$ parafermion theories [FZ]. The members of (3.23) may be also realized as the central charge and conformal weights of fusion $N+1$-state RSOS(p,p) lattice models [DJKMO], [BR] on the regime I/II critical line. Note that physical constraints

$$|m|\leq l, \quad m\equiv l \ (\mathrm{mod}\ 2)$$

for value of m in (3.23) are equivalent to a condition that "remainder term" in (3.22), namely, $6(l-|m|)$, must be belongs to $12\mathbf{Z}_+$.

Corollary 3.12. Let us fix the positive integers k, $p=1,2,\dots$ (the fusion level), j_1 and j_2 such that $0\leq j_1\leq k$, $0\leq j_2\leq k+l$.

Let $r_0 = p\left\{\dfrac{j_1-j_2}{p}\right\}$ be the unique interger determined by

$$0\leq r_0\leq p, \quad r_0\equiv \pm(j_1-j_2)\mathrm{mod}\ 2p. \tag{3.24}$$

Then we have

$$s(j_1,2,k) + s(r_0,2,p) - s(j_2,2,k+p) = \tag{3.25}$$

$$= c - 24\Delta + 12\,\frac{(j_1-j_2)(p+j_1-j_2)+r_0(p-r_0)}{2p},$$

where

$$c=\frac{3p}{p+2}\left(1-\frac{2(p+2)}{(k+2)(k+p+2)}\right),$$
$$\Delta=\frac{[(k+p+2)(j_1+1)-(k+2)(j_2+1)]^2-p^2}{4p(k+2)(k+p+2)}+\frac{r_0(p-r_0)}{2p(p+2)} \tag{3.26}$$

are the central charge and conformal weights of the fusion $(k+p+1)$-state $\mathrm{RSOS}(p,p)$ latice models [KP] on the regime III/IV critical line. It is easy to see that "remainder term" in (3.25) belongs to $12\mathbf{Z}_+$. Note also that the fusion $\mathrm{RSOS}(p,q)$ lattice models, obtained by fusing $p\times q$ blocks of face weights together, are related to coset conformal fields theories obtained by the Goddard-Kent-Olive (GKO) construction [GKO]. Namely, c and Δ in (3.26) are the central charge and conformal dimensions of conformal field theory, which corresponds to the coset pair [GKO]

$$\begin{array}{lccccc} & A_1 & \oplus & A_1 & \supset & A_1 \\ \text{levels} & k & & p & & k+p \end{array}$$

Thus the members of (3.26) are reduced to those of (3.11) if $p=1$ and of (3.15) if $p=2$.

§4. A_1-type dilogarithm identities.

As is well-known [Le2], the Rogers dilogarithm function $L(x)$ admits a continuation on all complex plane $\mathbf{C}$. Follow [Le], [KR] we define a function

$$\begin{aligned} L(x,\theta):&=-\frac{1}{2}\int_0^x\frac{\log(1-2x\cos\theta+x^2)}{x}dx+\frac{1}{4}\log|x|\cdot\log(1-2x\cos\theta+x^2)= \\ &=ReL(xe^{i\theta}),\quad x,\theta\in\mathbf{R} \end{aligned} \tag{4.1}$$

Our proof of Theorem 3.1 is based on a study of properties of the function $L(x,\theta)$.

Proposition 4.1. For all real $\varphi,\ \theta$ we have

$$\begin{aligned} L\Big(\Big(\frac{\sin\theta}{\sin\varphi}\Big)^2\Big)&=\pi^2\Big\{\overline{B}_2\left(\frac{\theta+\varphi}{\pi}\right)-\overline{B}_2\left(\frac{\varphi}{\pi}\right)-\overline{B}_2\left(\frac{\theta}{\pi}\right)+\frac{1}{6}\Big\}+ \\ &+2L\Big(-\frac{\sin(\varphi-\theta)}{\sin\theta},\varphi\Big)-2L\Big(-\frac{\sin\varphi}{\sin\theta},\varphi+\theta\Big). \end{aligned} \tag{4.2}$$

Before proving a Proposition 4.1 let us give the others useful properties of function (4.1) (compare with [Le2]).

Lemma 4.2.

$$(i)\quad L(x,0)=L(x),\quad L(-x,\varphi)=L(-x,\pi-\varphi) \tag{4.3}$$

$$(ii)\quad L(x,\varphi)=L(x,2\pi k\pm\varphi),\quad k\in\mathbf{Z} \tag{4.4}$$

$$(iii)\quad L(-1,\varphi)=\pi^2\overline{B}_2\left(\frac{\varphi}{2\pi}+\frac{1}{2}\right),$$

$$L(1,\varphi)=\pi^2\overline{B}_2\left(\frac{\varphi}{2\pi}\right) \tag{4.5}$$

$$(iv)\quad L(x,\varphi)+L(x^{-1},\varphi)=2\pi^2\overline{B}_2\left(\frac{\varphi}{2\pi}\right),\quad x>0$$

$$L(-x,\varphi)+L(-x^{-1},\varphi)=2\pi^2\overline{B}_2\left(\frac{\varphi}{2\pi}+\frac{1}{2}\right),\quad x<0 \tag{4.6}$$

$$(v)\quad L(0,\varphi)=0,\quad L(+\infty,\varphi)=2\pi^2\overline{B}_2\left(\frac{\varphi}{2\pi}\right),$$

$$L(-\infty,\varphi)=2\pi^2\overline{B}_2\left(\frac{\varphi+\pi}{2\pi}\right) \tag{4.7}$$

$$(vi)\quad L(2\cos\varphi,\varphi)=\pi^2\left\{\overline{B}_2\left(\frac{\varphi}{\pi}\right)+\frac{1}{12}\right\} \tag{4.8}$$

$$(vii)\quad L(x^n,n\varphi)=n\sum_{k=0}^{n-1}L\left(x,\ \varphi+\frac{2k\pi}{n}\right),\quad x\in\mathbf{R}_+,$$

$$L(x^n)=n\sum_{k=0}^{n-1}L\left(x\cdot\exp\frac{2k\pi i}{n}\right),\quad x\in(0,1). \tag{4.9}$$

More generally (Rogers' identity [Ro])

$$L(1-y^n)=\sum_{k=1}^{n}\sum_{l=1}^{n}\Big[L(\lambda_k/\lambda_l)-L(x_k\lambda_l)\Big],$$

where $\{x_k\}_{k=1}^n$ are the roots of the equation

$$1-y^n=\prod_{k=1}^{n}(1-\lambda_k x).$$

Proof. At first let us remind some properties of modified Bernoulli polynomials.

$$\begin{aligned}&\frac{d\overline{B}_n(x)}{dx}=n\overline{B}_{n-1}(x),\\ &\overline{B}_n(x)=\overline{B}_n(x+1),\quad \overline{B}_n(-x)=(-1)^n\overline{B}_n(x),\\ &\overline{B}_p(nx)=n\sum_{k=1}^{n}\overline{B}_p\left(x+\frac{k}{n}\right).\end{aligned} \tag{4.10}$$

Note that identities (4.5) follows from the Fourier expansion for $\overline{B}_2(x)$ (see (3.2)).

In order to prove the identity (4.2) let us differentiate LHS and RHS of the last one with respect to φ using the following formula

$$dL(x,\varphi)=\left\{-\frac{1}{4}\ \frac{\log(1-2x\cos\varphi+x^2)}{x}+\frac{1}{2}\log|x|\frac{x-\cos\varphi}{1-2x\cos\varphi+x^2}\right\}dx \tag{4.11}$$

$$+\left\{-\tan^{-1}\left(\frac{x\sin\varphi}{1-x\cos\varphi}\right)+\frac{1}{2}\log|x|\frac{x\sin\varphi}{1-2x\cos\varphi+x^2}\right\}d\varphi.$$

Acting in such a manner we find

$$\frac{d}{d\varphi}L\left(-\frac{\sin\varphi}{\sin\theta},\varphi+\theta\right)=$$

$$=\varphi+\frac{1}{2}\cot(\varphi+\theta)\log\left(\frac{\sin\varphi}{\sin\theta}\right)-\frac{1}{2}\cot\varphi\log\left(\frac{\sin(\varphi+\theta)}{\sin\theta}\right), \tag{4.12}$$

$$\frac{d}{d\varphi}L\left(-\frac{\sin\theta}{\sin\varphi},\varphi+\theta\right)=$$

$$=\theta-\frac{1}{2}\cot(\varphi+\theta)\log\left(\frac{\sin\varphi}{\sin\theta}\right)-\frac{1}{2}\cot\varphi\log\left(\frac{\sin(\varphi+\theta)}{\sin\theta}\right), \tag{4.13}$$

$$\text{if}\quad 0<\varphi,\quad \theta<\pi,\quad \varphi+\theta<\pi; \tag{4.13a}$$

$$\frac{d}{d\varphi}L\left(\frac{\sin\theta}{\sin\varphi}\cdot\frac{\sin(\theta+\psi)}{\sin(\varphi+\psi)}\right)=$$

$$=\frac{1}{2}\Big[\cot\varphi+\cot(\varphi+\psi)\Big]\log\left(\frac{\sin(\varphi-\theta)\sin(\varphi+\theta+\psi)}{\sin\theta\sin(\theta+\psi)}\right)-$$

$$-\frac{1}{2}\Big[\cot(\varphi-\theta)+\cot(\varphi+\theta+\psi)\Big]\log\left(\frac{\sin\varphi\sin(\varphi+\psi)}{\sin\theta\sin(\theta+\psi)}\right), \tag{4.14}$$

$$\text{if}\quad 0<\theta<\varphi<\pi,\quad 0<\psi<\pi,\quad \varphi+\theta+\psi<\pi. \tag{4.14a}$$

Further, let us use the reduction rules (4.3), (4.4) and (4.6) (compare with (1.4) and (1.5)) if the angles $\varphi,\ \theta$ (or $\varphi,\ \theta,\ \psi$) do not satisfy the condition (4.13a) (or (4.14a)). As a result one can obtain that a derivative of difference between LHS and RHS of (4.2) with respect to φ is equal to

$$2\pi\left\{\overline{B}_1\left(\frac{\varphi+\theta}{\pi}\right)-\overline{B}_1\left(\frac{\varphi}{\pi}\right)\right\}.$$

Integrating (see (4.10)), then taking $\varphi=0$ and using (4.5) to determine the integration constant, we obtain the equality (4.2).

It is easy to see that (4.8) follows from (4.2) when $\varphi+\theta=\pi$. In order to prove (4.9) let us differentiate LHS and RHS of (4.9) with respect to x and use a summation formula

$$\sum_{k=0}^{n-1}\frac{\exp i(\varphi+\frac{2\pi k}{n})}{1-x\exp i(\varphi+\frac{2\pi k}{n})}=\frac{nx^{n-1}\exp(in\varphi)}{1-x^n\exp(in\varphi)}.$$

Thus, the proofs of Proposition 4.1 and Lemma 4.2 are finished.

■

Proof of the Theorem 3.1 for the case $n=2$. If we substitute $\theta = m\varphi$ in (4.2) then obtain

$$L\left(\left(\frac{\sin m\varphi}{\sin\varphi}\right)^2\right) = \pi^2\left\{\overline{B}_2\left(\frac{(m+1)\varphi}{\pi}\right) - \overline{B}_2\left(\frac{m\varphi}{\pi}\right) - \overline{B}_2\left(\frac{\varphi}{\pi}\right) + \frac{1}{6}\right\} + \\ + 2L\left(-\frac{\sin((m-1)\varphi)}{\sin\varphi}, m\varphi\right) - 2L\left(-\frac{\sin m\varphi}{\sin\varphi}, (m+1)\varphi\right). \tag{4.15}$$

Futher let us introduce notation

$$f_m(\varphi) := 1 - \frac{Q_{m-1}(\varphi)Q_{m+1}(\varphi)}{Q_m^2(\varphi)} = \frac{1}{Q_m^2(\varphi)}.$$

Then using (4.15) we find

$$\sum_{m=1}^{r} L(f_m(\varphi)) = -2\left\{L\Big(-Q_r(\varphi), (r+2)\varphi\Big) - \frac{\pi^2}{6}\right\} + \tag{4.16}$$
$$+ \pi^2\left\{\overline{B}_2\left(\frac{(r+2)\varphi}{\pi}\right) - \frac{1}{6}\right\} + (r+2)\pi^2\left\{\frac{1}{6} - \overline{B}_2\left(\frac{\varphi}{\pi}\right)\right\} - \frac{\pi^2}{2}.$$

Now let us put $\varphi = \dfrac{(j+1)\pi}{r+2}$, $0 \le j \le r+1$. Then $Q_r(\varphi) = (-1)^j$ and it is clear from (4.3) and (4.4) that

$$L\Big((-1)^{j+1},\ (j+1)\pi\Big) = L(1) = \frac{\pi^2}{6}.$$

■

Note that polynomials $Q_m := Q_m(\varphi)$ satisfy the following recurrence relation

$$Q_m^2 = Q_{m-1}Q_{m+1} + 1, \quad Q_0 \equiv 1, \quad m \ge 1,$$

where as the polynomials $y_m := y_m(\varphi) = Q_{m-1}(\varphi)\cdot Q_{m+1}(\varphi)$ satisfy the following one

$$y_m^2 = (1 + y_{m-1})(1 + y_{m+1}), \quad y_0 \equiv 0, \quad m \ge 1.$$

Follow [Le2] we define a function $W(x,\varphi)$ by

$$W(x,\varphi) := W(x,\varphi,\theta) = L\left(\frac{\sin^2\theta}{\sin\varphi(\sin\varphi + x\sin(\varphi+\theta))}\right) + \\ + L\left(-\frac{x^2\sin\varphi + x\sin(\varphi-\theta)}{x\sin(\varphi+\theta) + \sin\varphi}\right) - L\left(\frac{x\sin(\varphi+\theta)}{x\sin(\varphi+\theta)+\sin\varphi}\right). \tag{4.17}$$

Note the following particular cases

$$W(0,\varphi)=L\left(\left(\frac{\sin\theta}{\sin\varphi}\right)^2\right),\quad W(-2\cos\theta,\varphi)=L\left(-\frac{\sin^2\theta}{\sin\varphi\sin(\varphi+2\theta)}\right),\tag{4.18}$$

$$W(1,\varphi)=L\left(\frac{\sin\theta\sin\frac{1}{2}\theta}{\sin\varphi\sin(\varphi+\frac{1}{2}\theta)}\right)+L\left(\frac{\sin(\frac{1}{2}\theta-\varphi)}{\sin(\frac{1}{2}\theta+\varphi)}\right)-L\left(\frac{\sin(\varphi+\theta)}{2\sin(\varphi+\frac{1}{2}\theta)\cos\frac{1}{2}\theta}\right),$$

$W(-1,\varphi,\theta)=W(1,\varphi,\pi+\theta)$.

Proposition 4.3. We have

$$W(x,\varphi)=2L(-x,\theta)+2L(-x_1,\varphi)-2L(-x_2,\varphi+\theta)+$$
$$+\pi^2\left\{\overline{B}_2\left(\frac{\varphi+\theta}{\pi}\right)-\overline{B}_2\left(\frac{\varphi}{\pi}\right)-\overline{B}_2\left(\frac{\theta}{\pi}\right)+\frac{1}{6}\right\},\tag{4.19}$$

where $\quad x_1=\dfrac{x\sin\varphi+\sin(\varphi-\theta)}{\sin\theta}\quad$ and $\quad x_2=\dfrac{x\sin(\varphi+\theta)+\sin\varphi}{\sin\theta}$.

A proof of Proposition 4.3 may be obtained by the same manner as that of Proposition 4.1.

■

Note that x_2 is obtained from x_1 by replacing φ by $\varphi+\theta$. The angular parameter in (4.19) also increases in this way and the terms $L(-x,\varphi)$ and $L(-x_2,\varphi+\theta)$ have opposite signs. So if we substitute successively the angles $\varphi,\varphi+\theta,\ldots,\varphi+r\theta$ instead of φ in (4.19) and after this will add all results together, we will obtain the following generalisation of (4.16):

$$\sum_{k=0}^{r}W(x,\varphi+k\theta)=2(r+1)L(-x,\theta)+2L(-x_1,\varphi)-2L(-x_{r+2},\varphi+(r+1)\theta)+$$
$$+\pi^2\left\{\overline{B}_2\left(\frac{\varphi+(r+1)\theta}{\pi}\right)-\overline{B}_2\left(\frac{\varphi}{\pi}\right)\right\}+(r+1)\pi^2\left\{\frac{1}{6}-\overline{B}_2\left(\frac{\theta}{\pi}\right)\right\},\tag{4.20}$$

where $\quad x_{n+1}:=\dfrac{x\sin(\varphi+n\theta)+\sin(\varphi+(n-1)\theta)}{\sin\theta},\quad 0\le n\le r+2.$

Now let us take $\varphi=\theta$ in (4.20). Then we find $x_1=x$, so that (4.20) becomes

$$\sum_{k=1}^{r+1}W(x,k\theta)=2(r+2)L(-x,\theta)+(r+2)\pi^2\left\{\frac{1}{6}-\overline{B}_2\left(\frac{\theta}{\pi}\right)\right\}-\frac{\pi^2}{3}+\tag{4.21}$$
$$+\pi^2\left\{\overline{B}_2\left(\frac{(r+2)\theta}{\pi}\right)-\frac{1}{6}\right\}-2\left(L(-x_{r+2},(r+2)\theta)-\frac{\pi^2}{6}\right),$$

where $\quad x_{n+1}:=x_{n+1}(x,\theta)=\dfrac{x\sin(n+1)\theta+\sin n\theta}{\sin\theta}.$

Note the following particular cases ($0 \le n \le r+2$)

$$x_{n+1}(0,\theta) = \frac{\sin n\theta}{\sin\theta}, \quad x_{n+1}(-2\cos\theta,\theta) = -\frac{\sin(n+2)\theta}{\sin\theta},$$

$$x_{n+1}(1,\theta) = \frac{\sin\left(\frac{1}{2}(2n+1)\theta\right)}{\sin\frac{1}{2}\theta}, \quad x_{n+1}(-1,\theta) = x_{n+1}(1,\theta+\pi).$$

One can show that an identity (4.21) is reduced to (4.2) if $x = -2\cos\theta$ (or $x = 0$). Now assume $x \ne -2\cos\theta$ and take $\theta = \dfrac{(j+1)\pi}{r+2}$ in (4.21). Then we find $x_{r+2} = (-1)^j$, so that (4.21) becomes

$$\sum_{k=1}^{r+1} W(x,k\theta) = 2(r+2)L(-x,\theta) + \frac{\pi^2}{6}(1+s(j,2,r)), \tag{4.22}$$

where $\quad s(j,2,r) = \dfrac{3r}{r+2} + 6\,\dfrac{j(r-j)}{r+2}\quad$ (see (3.7)).

Finally let us take $x = \pm 1$ in (4.22). After some manipulations we obtain the following result.

Proposition 4.4. Let functions $W(\pm 1,\theta)$ are defined by (4.18) and $\theta = \dfrac{(j+1)\pi}{r+2}$. Then we have

$$\sum_{k=1}^{r+1} W(-1,k\theta) = \frac{\pi^2}{6}\left\{2r+2-\frac{3(j+1)^2}{r+2}\right\},$$
$$\sum_{k=1}^{r+1} W(+1,k\theta) = \frac{\pi^2}{6}\left\{2-r-\frac{3(j+1)^2}{r+2}+6j\right\}. \tag{4.23}$$

■

Now we propose a generalisation of (4.16). Given a rational number p and decomposition of p into the continued fraction

$$p = [b_r, b_{r-1}, \dots, b_1, b_0] = b_r + \cfrac{1}{b_{r-1} + \cfrac{1}{\dots + \cfrac{1}{b_1 + \cfrac{1}{b_0}}}}. \tag{4.24}$$

We will assume that $b_i > 0$ if $0 \le i < r$ and $b_r \in \mathbf{Z}$. Using the decomposition (4.24) we define the set of integers y_i and m_i:

$$y_{-1} = 0, \quad y_0 = 1, \quad y_1 = b_0, \dots, y_{i+1} = y_{i-1} + b_i y_i, \quad 0 \le i \le r, \tag{4.25}$$
$$m_0 = 0, \quad m_1 = b_0, \quad m_{i+1} = |b_i| + m_i, \quad 0 \le i \le r.$$

It is clear that $p = \dfrac{y_{r+1}}{y_r}$ and

$$\frac{y_{i+1}}{y_i} = p_i := b_i + \cfrac{1}{b_{i-1} + \cfrac{1}{\cdots + \cfrac{1}{b_1 + \cfrac{1}{b_0}}}}, \quad 0 \le i \le r.$$

The following sequences of integers were first introduced by Takahashi and Suzuki [TS]

$$\begin{aligned} &r(j) = i, \quad \text{if} \quad m_i \le j < m_{i+1}, \quad 0 \le i \le r, \\ &n_j = y_{i-1} + (j - m_i) y_i, \quad \text{if} \quad m_i \le j < m_{i+1} + \delta_{i,r}, \quad 0 \le i \le r. \end{aligned}$$

Finally we define a dilogarithm sum of "fractional level p":

$$\sum_{j=1}^{m_{r+1}} (-1)^{r(j)} L\left(\left(\frac{\sin y_{r(j)}\theta}{\sin(n_j + y_{r(j)})\theta} \right)^2 \right) := (-1)^r \frac{\pi^2}{6} s(k, 2, p), \tag{4.26}$$

where $\theta = \dfrac{(k+1)\pi}{y_{r+1} + 2y_r}$.

The dilogarithm sum (4.26) (in the case $k = 0$) was considered at first in [KR], where its interpretation as a low-temperature asymptotic of the entropy for the XXZ Heisenberg model was given.

Proposition 4.5. We have

$$\begin{aligned} &(i) \quad s(0, 2, p) = \frac{3p}{p+2}, \\ &(ii) \quad s(k, 2, p) = \frac{3p}{p+2} - 6\frac{k(k+2)}{p+2} + 6\mathbf{Z}. \end{aligned} \tag{4.27}$$

Proof. We start with

Lemma 4.6. Given an integer σ such that $m_i < \sigma \le m_{i+1}$. Then for any $\theta \in \mathbf{R}$ we have

$$\begin{aligned} \sum_{j=m_i}^{\sigma-1} L\left(\left(\frac{\sin y_{r(j)}\theta}{\sin(n_j + y_{r(j)})\theta} \right)^2 \right) &:= (\sigma - m_i)\pi^2 \left\{ \frac{1}{6} - \overline{B}_2\left(\frac{y_i \theta}{\pi} \right) \right\} + \\ &+ \pi^2 \left\{ \overline{B}_2\left(\frac{(n_{\sigma-1} + 2y_i)\theta}{\pi} \right) - \overline{B}_2\left(\frac{(y_{i-1} + y_i)\theta}{\pi} \right) \right\} + \\ &+ 2L\left(-\frac{\sin y_{i-1}\theta}{\sin y_i \theta}, (y_i + y_{i-1})\theta \right) - 2L\left(-\frac{\sin(n_{\sigma-1} + y_i)\theta}{\sin y_i \theta}, (n_{\sigma-1} + 2y_i)\theta \right). \end{aligned} \tag{4.28}$$

A proof of Lemma 4.6 follows from Proposition 4.1.

■

From (4.28) one can easily deduce the following generalisation of (4.16)

Corollary 4.7. Given an integer σ, $\sigma < m_i \le m_{i+1}$. Then

$$\sum_{j=0}^{\sigma-1} L\left(\left(\frac{\sin y_{r(j)}\theta}{\sin(n_j+y_{r(j)})\theta}\right)^2\right) := \pi^2\sum_{j=0}^{i-1}(-1)^j b_j\left\{\frac{1}{6}-\overline{B}_2\left(\frac{y_j\theta}{\pi}\right)\right\}+$$
$$+(-1)^i(\sigma-m_i)\pi^2\left\{\frac{1}{6}-\overline{B}_2\left(\frac{y_i\theta}{\pi}\right)\right\}+(-1)^i\pi^2\overline{B}_2\left(\frac{(n_{\sigma-1}+2y_i)\theta}{\pi}\right)-$$
$$-2\pi^2\sum_{j=0}^{i-1}(-1)^j\overline{B}_2\left(\frac{(y_{j-1}+y_j)\theta}{\pi}\right)-\pi^2\overline{B}_2\left(\frac{\theta}{\pi}\right)+ \tag{4.29}$$
$$+2\sum_{j=0}^{i-1}(-1)^{j+1}\left\{L\left(-\frac{\sin y_{j-1}\theta}{\sin y_j\theta},(y_j+y_{j-1})\theta\right)+L\left(-\frac{\sin y_j\theta}{\sin y_{j-1}\theta},(y_j+y_{j-1})\theta\right)\right\}+$$
$$+(-1)^{i+1}2L\left(-\frac{\sin(n_{\sigma-1}+y_i)\theta}{\sin y_i\theta},(n_{\sigma-1}+2y_i)\theta\right).$$

In order to go further we must compute the last sum in (4.29). Such computation is based on the following result.

Lemma 4.8. Given the real numbers φ and θ, let us define

$$\epsilon(\varphi,\theta)=\{\frac{\varphi}{2\pi}+\frac{1}{2}\}+\{\frac{\theta}{2\pi}+\frac{1}{2}\}-\{\frac{\varphi}{2\pi}\}-\{\frac{\theta}{2\pi}\}\pmod 2, \tag{4.30}$$

where $\{x\}=x-[x]$ be a fractional part of $x\in\mathbf{R}$.

Then we have

$$L\left(-\frac{\sin\varphi}{\sin\theta},\varphi+\theta\right)+L\left(-\frac{\sin\theta}{\sin\varphi},\varphi+\theta\right)=2\pi^2\overline{B}_2\left(\frac{\varphi+\theta+\epsilon(\varphi,\theta)\pi}{2\pi}\right). \tag{4.31}$$

A proof of Lemma 4.8 follows from (4.6).

■

Now we are ready to finish a proof of Proposition 4.5. Namely from (4.29) and (4.31) it follows that $s(k,2,p)\in\mathbf{Q}$, $(y_{r+1}+2y_r)\cdot s(k,2,p)\in\mathbf{Z}$, and

$$s(0,2,p)=\frac{3p}{p+2}.$$

Finally, we observe that (4.27) follows from (4.29) and (4.31) if we replace all modified Bernoulli polynomials by ordinary ones.

■

Proposition 4.9. For all positive $p\in\mathbf{Q}$ the remainder term in (4.27) lies in $6\mathbf{Z}_+$. More exactly, given a positive $p\in\mathbf{Q}$ we define a set of integers $\{s_k\}$, $k=0,1,2,\dots$ such that

$$\left[\frac{j+1}{p+2}\right]=k \ \text{ iff } \ s_k\le j<s_{k+1},\ s_0:=0.$$

Further, let us define a function

$$t(j) := t(j,p) = (2k+1)j + k - 2\sum_{a=0}^{k} s_a \quad \text{iff} \quad s_k \le j < s_{k+1}.$$

Then we have

$$s(j,2,p) = \frac{3p}{p+2} - \frac{6j(j+2)}{p+2} + 6t(j,p).$$

■

It is clear that $t(j,p) \in \mathbf{Z}_+$.

Corollary 4.10. Let us fix the positive integers $l = 1,2,3,\dots$ (the fusion level), $p > q$, j_1 and j_2. Then

$$s\left(j_1, 2, \frac{ql}{p-q} - 2\right) + s(r_0, 2, l) - s\left(j_2, 2, \frac{pl}{p-q} - 2\right) = c - 24\Delta + 6\mathbf{Z}, \tag{4.32}$$

where $r_0 = l \cdot \left\{\dfrac{j_1 - j_2}{l}\right\}$ and

$$\begin{aligned} c &= \frac{3l}{l+2}\left(1 - \frac{2(l+2)(p-q)^2}{l^2pq}\right), \\ \Delta &= \frac{[p(j_1+1) - q(j_2+1)]^2 - (p-q)^2}{4lpq} + \frac{r_0(l-r_0)}{2l(l+2)} \end{aligned} \tag{4.33}$$

are the central charge and conformal dimensions of RCFT, which corresponds to the coset pair [GKO]

$$\begin{array}{lccccc} & A_1 & \oplus & A_1 & \supset & A_1 \\ \text{levels} & \dfrac{ql}{p-q} - 2 & & l & & \dfrac{pl}{p-q} - 2 \end{array}$$

§5. Proof of Theorem 3.1.

We start with a generalization of identity (4.2).

Proposition 5.1. For all real φ, ψ and θ we have

$$\begin{aligned} & L\left(\frac{\sin\theta}{\sin\varphi} \cdot \frac{\sin(\theta+\psi)}{\sin(\varphi+\psi)}\right) = \\ & = \pi^2\left\{\overline{B}_2\left(\frac{\theta+\varphi+\psi}{\pi}\right) - \overline{B}_2\left(\frac{2\theta+\psi+\overline{\epsilon}(\theta,\theta+\psi)\pi}{2\pi}\right) - \right. \end{aligned} \tag{5.1}$$

$$
\begin{aligned}
&-\overline{B}_2\biggl(\frac{2\varphi+\psi+\overline{\epsilon}(\varphi,\varphi+\psi)\pi}{2\pi}\biggr)+\frac{1}{6}\biggr\}+\\
&+L\biggl(-\frac{\sin(\varphi-\theta)}{\sin\theta},\varphi\biggr)+L\biggl(-\frac{\sin(\varphi-\theta)}{\sin(\theta+\psi)},\varphi+\psi\biggr)-\\
&-L\biggl(-\frac{\sin\varphi}{\sin(\theta+\psi)},\varphi+\theta+\psi\biggr)-L\biggl(-\frac{\sin(\varphi+\psi)}{\sin\theta},\varphi+\theta+\psi\biggr),
\end{aligned}
$$

where $\overline{\epsilon}(\varphi,\theta)=1-\epsilon(\varphi,\theta)$ and $\epsilon(\varphi,\theta)$ is defined by (4.31).

Proof. First of all we consider the case when $0<\varphi+\theta+\psi<\pi$ and $\varphi,\theta,\psi>0$. In this case $\overline{\epsilon}(\theta,\theta+\psi)=\overline{\epsilon}(\varphi,\varphi+\psi)=0$ and one can use the identities (4.12)-(4.14) in order to show that a derivative of difference between LHS and RHS of (5.1) with respect to φ is equal to

$$
2\pi\left\{B_1\left(\frac{\theta+\varphi+\psi}{\pi}\right)-B_1\left(\frac{2\varphi+\psi}{2\pi}\right)\right\}=2\theta+\psi.
$$

Integrating (see (4.10)) we find that the difference between LHS and RHS of (5.1) is a function $c(\theta,\psi)$ which does not depend on φ. In order to find $c(\theta,\psi)$ let us take $\varphi=\theta$ in (5.1). After this substitution we obtain an equality

$$
\begin{aligned}
\frac{\pi^2}{6}=&\frac{1}{2}(2\theta+\psi)^2+c(\theta,\psi)-\\
&-L\left(-\frac{\sin\theta}{\sin(\theta+\psi)},2\theta+\psi\right)-L\left(-\frac{\sin(\theta+\psi)}{\sin\theta},2\theta+\psi\right).
\end{aligned}
$$

Comparing the last equality with (4.31) (in our case $\epsilon(\theta,\theta+\psi)=1$) we find $c(\theta,\psi)=0$. In general case we use the reduction rules (4.3), (4.4) and (4.6) and the following properties of function $\epsilon(\varphi,\theta)$:

$$
\begin{aligned}
&\epsilon(\theta,\theta)=1, &&\epsilon(\theta,\varphi)=\epsilon(\varphi,\theta),\\
&\epsilon(\theta,\pi+\theta)=0, &&\epsilon(\varphi,-\theta)=\epsilon(\varphi,\pi+\theta),\\
&\epsilon(\theta,\pi-\theta)=1, &&\epsilon(-\varphi,-\theta)=\epsilon(\varphi,\theta).
\end{aligned}
$$

■

Corollary 5.2. We have

$$
L\left(\frac{\sin(\varphi+\theta)}{\sin\theta},\varphi\right)+L\left(\frac{\sin(\varphi+\theta)}{\sin\varphi},\theta\right)=2\pi^2\biggl\{\overline{B}_2\left(\frac{\varphi+\theta+\overline{\epsilon}(\varphi,\theta)\pi}{2\pi}\right)+\frac{1}{12}\biggr\}. \tag{5.2}
$$

Proof. Take $\psi=-\theta-\varphi$ in (5.1).

■

Let us continue a proof of Theorem 3.1 and take a specialisation $\theta\to k\varphi$, $\varphi\to(m+k)\varphi$ and $\psi\to(n-2k)\varphi$ in (5.1). Then we obtain

$$
L\biggl(\frac{\sin k\varphi}{\sin(m+k)\varphi}\cdot\frac{\sin(n-k)\varphi}{\sin(m+n-k)\varphi}\biggr)=
$$

$$= \pi^2 \Big\{ \overline{B}_2\Big(\frac{(m+n)\varphi}{\pi}\Big) - \overline{B}_2\Big(\frac{n\varphi + \overline{\epsilon}(k\varphi,(n-k)\varphi)\pi}{2\pi}\Big) -$$
$$- \overline{B}_2\Big(\frac{(n+2m)\varphi + \overline{\epsilon}((m+k)\varphi,(n-k)\varphi)\pi}{2\pi}\Big) + \frac{1}{6}\Big\} +$$
$$+ L\Big(-\frac{\sin m\varphi}{\sin k\varphi}, (m+k)\varphi\Big) + L\Big(-\frac{\sin m\varphi}{\sin(n-k)\varphi}, (m+n-k)\varphi\Big) -$$
$$- L\Big(-\frac{\sin(m+k)\varphi}{\sin(n-k)\varphi}, (m+n)\varphi\Big) - L\Big(-\frac{\sin(m+n-k)\varphi}{\sin k\varphi}, (m+n)\varphi\Big).$$

Consequently, after summation we obtain

$$\sum_{k=1}^{n-1}\sum_{m=1}^{r} L\Big(\frac{\sin k\varphi}{\sin(m+k)\varphi}\cdot\frac{\sin(n-k)\varphi}{\sin(m+n-k)\varphi}\Big) = 2\sum_{k=1}^{n-1}\sum_{m=1}^{r} L\Big(-\frac{\sin m\varphi}{\sin k\varphi}, (m+k)\varphi\Big) -$$
$$- 2\sum_{k=1}^{n-1}\sum_{m=1}^{r} L\Big(-\frac{\sin(m+n-k)\varphi}{\sin k\varphi}, (m+n)\varphi\Big) + \pi^2\Sigma_3 =$$
$$= 2\sum_{k=1}^{n-1}\sum_{m=1}^{n-k} L\Big(-\frac{\sin m\varphi}{\sin k\varphi}, (m+k)\varphi\Big) - 2\sum_{k=1}^{n-1}\sum_{m=1}^{n-k} L\Big(-\frac{\sin(r+m)\varphi}{\sin k\varphi}, (m+r+k)\varphi\Big) +$$
$$+ \pi^2\Sigma_3 = 2\Sigma_1 - 2\Sigma_2 + \pi^2\Sigma_3, \quad \text{where}$$

$$\Sigma_3 := \sum_{k=1}^{n-1}\sum_{m=1}^{r} \Big\{ \overline{B}_2\Big(\frac{(m+n)\varphi}{\pi}\Big) - \overline{B}_2\Big(\frac{n\varphi + \overline{\epsilon}(k\varphi,(n-k)\varphi)\pi}{2\pi}\Big) -$$
$$- \overline{B}_2\Big(\frac{(n+2m)\varphi + \overline{\epsilon}((m+k)\varphi,(n-k)\varphi)\pi}{2\pi}\Big) + \frac{1}{6}\Big\}. \tag{5.3}$$

At first, let us consider the sum

$$2\Sigma_1 := 2\sum_{k=1}^{n-1}\sum_{m=1}^{n-k} L\Big(-\frac{\sin m\varphi}{\sin k\varphi}, (m+k)\varphi\Big) = 2\sum_{p=1}^{[\frac{n}{2}]} L(-1, 2p\varphi) + \tag{5.4}$$
$$+ 2\sum_{p=3}^{n}\sum_{k=1}^{[\frac{p-1}{2}]} \Big\{ L\Big(-\frac{\sin(p-k)\varphi}{\sin k\varphi}, p\varphi\Big) + 2L\Big(-\frac{\sin k\varphi}{\sin(p-k)\varphi}, p\varphi\Big)\Big\} =$$
$$= 2\sum_{p=3}^{n}\sum_{k=1}^{[\frac{p-1}{2}]} 2\pi^2\overline{B}_2\left(\frac{p\varphi + \overline{\epsilon}((p-k)\varphi, k\varphi)\pi}{2\pi}\right) + 2\sum_{p=1}^{[\frac{n}{2}]} \pi^2\overline{B}_2\left(\frac{p\varphi}{\pi} + \frac{1}{2}\right).$$

Secondly, in order to compute the sum $2\Sigma_2$, let us remind that $\varphi = \dfrac{(j+1)\pi}{n+r}$. Hence $\sin(m+r)\varphi = \sin\dfrac{(m+r)(j+1)\pi}{n+r} = (-1)^j \sin(n-m)\varphi$ and consequently (see (4.3) and (4.4))

$$L\Big(-\frac{\sin(r+m)\varphi}{\sin k\varphi},(m+r+k)\varphi\Big)=$$
$$=L\Big((-1)^{j+1}\frac{\sin(n-m)\varphi}{\sin k\varphi},(j+1)\pi-(n-m-k)\varphi\Big)=$$
$$=L\Big(-\frac{\sin(m-n)\varphi}{\sin k\varphi},(m+k-n)\varphi\Big).$$

So we have

$$2\Sigma_2:=2\sum_{k=1}^{n-1}\sum_{m=1}^{n-k}L\Big(-\frac{\sin(r+m)\varphi}{\sin k\varphi},(m+r+k)\varphi\Big)= \tag{5.5}$$
$$=\sum_{k=1}^{n-1}\sum_{m=1}^{n-k}L\Big(-\frac{\sin(m-n)\varphi}{\sin k\varphi},(m+k-n)\varphi\Big)=2\sum_{p=1}^{[\frac{n-1}{2}]}L(2\cos p\varphi,p\varphi)+\frac{(n-1)\pi^2}{3}+$$
$$+2\sum_{p=3}^{n-1}\sum_{k=1}^{[\frac{p-1}{2}]}\Big\{L\Big(\frac{\sin p\varphi}{\sin k\varphi},(p-k)\varphi\Big)+L\Big(\frac{\sin p\varphi}{\sin(p-k)\varphi},k\varphi\Big)\Big\}=\frac{(n-1)\pi^2}{3}+$$
$$+2\sum_{p=3}^{n-1}\sum_{k=1}^{[\frac{p-1}{2}]}2\pi^2\Big\{\overline{B}_2\left(\frac{p\varphi+\overline{\epsilon}(k\varphi,(p-k)\varphi)\pi}{2\pi}\right)+\frac{1}{12}\Big\}+2\sum_{p=1}^{[\frac{n}{2}]}\pi^2\Big\{\overline{B}_2\left(\frac{p\varphi}{\pi}\right)+\frac{1}{12}\Big\}.$$

Let us sum up our computations. First of all we proved that $s(j,n,k)\in\mathbf{Q}$. Secondly, in order to compute the dilogarith sum $s(j,n,k)$ modulo$\mathbf{Z}$ we may replace all modified Bernoulli polinomials appearing in (5.3)-(5.5) by ordinary ones. After some bulky calculations we obtain (3.5), except positivity of a remainder term in (3.5). Finally, in order to obtain the exact formulae (3.3) and (3.4) we are based on the properties of Dedekind sums [Ra]. Details will appear elsewhere.

■

Now we propose a generalisation of (4.27). For this goal let us define the following function

$$L_k(\theta,\varphi):=2L\left(\frac{\sin\theta\cdot\sin k\theta}{\sin\varphi\cdot\sin(\varphi+(k-1)\theta)}\right)-\sum_{j=0}^{k-1}L\left(\left(\frac{\sin\theta}{\sin(\varphi+j\theta)}\right)^2\right). \tag{5.6}$$

Lemma 5.3. We have

$$L_k(\theta,\varphi)=2L\left(-\frac{\sin(\varphi-\theta)}{\sin k\theta},\ \varphi+(k-1)\theta\right)-2L\left(-\frac{\sin\varphi}{\sin k\theta},\ \varphi+k\theta\right)+\pi^2\mathbf{Q}.$$

■

Now let $p\in\mathbf{Q}$ and consider a decomposition of p/k into continued fraction

$$\frac{p}{k}=b_r+\cfrac{1}{b_{r-1}+\cfrac{1}{\cdots+\cfrac{1}{b_1+\cfrac{1}{b_0}}}}, \tag{5.7}$$

where $b_i \in \mathbf{N}$, $\ 0 \leq i \leq r-1$ and $b_r \in \mathbf{Z}$. Using the decomposition (5.7) we define (compare with (4.25)):

$$\begin{aligned}
&y_{-1} = 0, \quad y_0 = 1, \quad y_1 = b_0, \ldots, y_{i+1} = y_{i-1} + b_i y_i, \\
&m_0 = 0, \quad m_1 = b_0, \quad m_{i+1} = |b_i| + m_i, \\
&r(j) := r_k(j) = i, \quad \text{if} \quad km_i \leq j < km_{i+1} + \delta_{i,r}, \\
&n_j := n_k(j) = ky_{i-1} + (j - km_i)y_i, \quad \text{if} \quad km_i \leq j < km_{i+1} + \delta_{i,r},
\end{aligned}$$

where $0 \leq i \leq r$.

Finally, we consider the following dilogarithm sum

$$\sum_{j=1}^{km_{r+1}} (-1)^{r(j)} L_k\Big(y_{r(j)}\theta, \ (n_j + y_{r(j)})\theta\Big) = (-1)^r \frac{\pi^2}{6} s(l, k+1, p), \tag{5.8}$$

where $\theta = \dfrac{(l+1)\pi}{ky_{r+1} + (k+1)y_r}$.

Proposition 5.4. We have

$$(i) \quad s(0, k+1, p) := c_k = \frac{3(p+1-k)}{p+k+1}, \quad k \geq 1, \tag{5.8}$$

$$(ii) \quad s(l, k+1, p) = c_k - \frac{6k\ l(l+2)}{p+k+1} + 6\mathbf{Z}, \tag{5.9}$$

(iii) if $k = 1$ or 2, then the remainder term in (5.9) lies in $6\mathbf{Z}_+$.

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